

CYCLIC AND FINITE SURGERIES ON MONTESINOS KNOTS

KAZUHIRO ICHIHARA AND IN DAE JONG

ABSTRACT. We give a complete classification of the Dehn surgeries on Montesinos knots which yield manifolds with cyclic or finite fundamental groups.

1. INTRODUCTION

A *Dehn surgery* on a knot K in a 3-manifold M is an operation to create a new 3-manifold from M and K as follows: Remove an open tubular neighborhood of K , and glue a solid torus back. By gluing a solid torus back as it was, the surgery gives the original manifold again. So such a surgery is called *trivial*, and we will ignore it in general.

On knots in the 3-sphere S^3 , it is an interesting problem to determine and classify all non-trivial Dehn surgeries which produce 3-manifolds with cyclic or finite fundamental groups, which we call *cyclic surgeries* / *finite surgeries* respectively.

As part of the Hyperbolic Dehn Surgery Theorem, Thurston [23] established that there are finitely many cyclic and finite surgeries. In fact, Culler, Gordon, Luecke and Shalen [4] (respectively, Boyer and Zhang [3]) proved there are at most three cyclic (resp., five finite) surgeries. Furthermore, it is conjectured that knots admitting cyclic (resp., finite) surgeries are doubly primitive (resp., primitive/Seifert fibered) knots as introduced by Berge [1] (resp., Dean [5]). See [13, Problem 1.77] for more information.

Cyclic and finite surgeries have been studied extensively for some classes of knots. For example, it was shown by Delman and Roberts in [8] that no hyperbolic alternating knot admits a cyclic or finite surgery.

One of the other well-known classes of knots, containing non-alternating ones, is the Montesinos knots. A *Montesinos knot* is defined as a knot admitting a diagram obtained by putting rational tangles together in a circle. See Figure 1 for instance. In particular, a Montesinos knot K is called a (a_1, a_2, \dots, a_n) -pretzel knot if the rational tangles in K are of the form $1/a_1, 1/a_2, \dots, 1/a_n$.

In this paper, based on studies by Delman [7] and Mattman [15], we give a complete classification of cyclic / finite surgeries on Montesinos knots as follows.

Theorem 1. *Let K be a hyperbolic Montesinos knot. If K admits a non-trivial cyclic surgery, then K must be equivalent to the $(-2, 3, 7)$ -pretzel knot and the surgery slope is 18 or 19. If K admits a non-trivial acyclic finite surgery, then K must be equivalent to either the $(-2, 3, 7)$ -pretzel knot and the surgery slope is 17, or the $(-2, 3, 9)$ -pretzel knot and the surgery slope is 22 or 23.*

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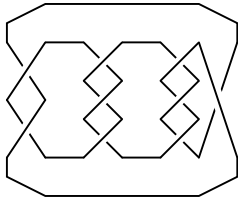


FIGURE 1. A diagram of a Montesinos knot

As a direct corollary, together with the result by Wu [25], we have the following.

Corollary 2. *Let K be a hyperbolic arborescent knot. If K admits a non-trivial cyclic surgery, then K must be equivalent to the $(-2, 3, 7)$ -pretzel knot and the surgery slope is 18 or 19. If K admits a non-trivial acyclic finite surgery, then K must be equivalent to either the $(-2, 3, 7)$ -pretzel knot and the surgery slope is 17, or the $(-2, 3, 9)$ -pretzel knot and the surgery slope is 22 or 23. \square*

Recently, using Khovanov homology, it was shown in [24, Theorem 7.5] that the $(-2, p, p)$ -pretzel knot does not admit finite surgeries for $p \in \{5, 7, \dots, 25\}$.

Very recently, Futer, Ishikawa, Kabaya, Mattman, and Shimokawa [9] obtained, independently, a complete classification of finite surgeries on $(-2, p, q)$ -pretzel knots with odd positive integers p and q .

Remark 1. It is already known which Montesinos knots are non-hyperbolic. If a Montesinos knot is equivalent to one consisting of at most two rational tangles, then it actually is a two-bridge knot. Menasco [16] showed that the non-hyperbolic two-bridge knots are the $(2, p)$ -torus knots. The only other non-trivial non-hyperbolic Montesinos knots are the $(-2, 3, 3)$ - and $(-2, 3, 5)$ -pretzel knots, which are actually the $(3, 4)$ - and $(3, 5)$ -torus knots, respectively. This was originally shown by Oertel [19, Corollary 5] as well as in an unpublished monograph [2] by Bonahon and Siebenmann. The cyclic and finite surgeries of torus knots have been completely classified by Moser [17].

To prove Theorem 1, we will prepare two propositions, Propositions 3 and 4, which will be shown in Sections 2 and 3 respectively. Then, in the last section, Theorem 1 will be proved from these propositions together with a result of Mattman [15].

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2. CYCLIC/FINITE SURGERIES AND THE ALEXANDER POLYNOMIALS

In this section, we prove the following proposition.

Proposition 3. *Let K be a hyperbolic Montesinos knot admitting a non-trivial cyclic or finite surgery. Then K is equivalent to a $(-1, 2n, p, q)$ -pretzel knot, where n is a non-zero integer and p, q are odd positive integers with $3 \leq p \leq q$. Furthermore all non-zero coefficients of the Alexander polynomial for K are ± 1 .*

Proof. Suppose that a hyperbolic Montesinos knot K admits a non-trivial cyclic or finite surgery. Then Delman showed in [6, 7] that K must be equivalent to either a $(-2l, p, q)$ -pretzel knot, a $(-1, 2n, p, q)$ -pretzel knot or a $(-1, -1, 2m, p, q)$ -pretzel knot with an integer n , integers $l, m > 1$ and odd positive integers p, q ($3 \leq p \leq q$). Also see [26, Section 2, Section 3]. Actually Delman showed that any Montesinos knot except for those listed above admits an essential lamination in its exterior which survives all non-trivial Dehn surgeries. *Essential laminations* were introduced by Gabai and Oertel in [11] and, actually, they showed that if a 3-manifold admits an essential lamination, then its universal cover must be the 3-space \mathbb{R}^3 . In particular its fundamental group is not cyclic or finite. See [11] for the precise definition.

By virtue of Delman's result, in order to prove Proposition 3, it suffices to show that the first and the third types of pretzel knots described above cannot have cyclic or finite surgeries. Note here that a $(-2, p, q)$ -pretzel knot (the case $l = 1$ in the first) is equivalent to a $(-1, 2, p, q)$ -pretzel knot (the case $n = 1$ in the second). Also a $(-1, -1, 2, p, q)$ -pretzel knot (the case $m = 1$ in the third) is equivalent to a $(-1, -2, p, q)$ -pretzel knot (the case $n = -1$ in the second). Thus, excluding overlaps, we are assuming $l, m \neq 1$.

Among the classes of knots described above, the first one was already studied by Mattman in [15]. He actually showed in [15, Theorem 1.1 and 1.2] that any $(-2l, p, q)$ -pretzel knot with $l > 1$ and odd positive integers p, q ($3 \leq p \leq q$) has neither cyclic surgeries nor finite surgeries.

Thus, in the following, we focus on the third class above. We here use the following strong result by Ni, [18, Corollary 1.3], established by using the Heegaard Floer homology theory: If a knot in S^3 admits a cyclic or finite surgery, then it must be a fibered knot. Actually he showed that a knot K in S^3 must be fibered if K admits a surgery yielding an L-space. Here a rational homology sphere Y is called an *L-space* if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$. In fact, any 3-manifold with a cyclic or finite fundamental group is an L-space, as is shown in [21, Proposition 2.3].

Now, the next claim, together with the result by Ni, imply the first conclusion of Proposition 3.

Claim 1. *Let $m > 1$ and p, q be odd positive integers ($p \leq q$). The $(-1, -1, 2m, p, q)$ -pretzel knot is not fibered.*

Proof. We just apply the algorithm given in [10, Theorem 6.7]. Here we include only an outline, assuming that the reader is rather familiar with [10, Theorem 6.7]. Please see [10] for details.

Let K be a $(-1, -1, 2m, p, q)$ -pretzel knot with an integer $m > 1$ and odd positive integers p, q ($p \leq q$). We start to apply the algorithm in [10, Theorem 6.7] with $n_1 = -1$, $n_2 = -1$, $n_3 = 2m$, $n_4 = p$, $n_5 = q$. After a cyclic permutation, the surface R obtained by applying Seifert's algorithm is of type II in [10, TYPE II.6.5] with $m_1 = -1$, $m_{11} = 2m$, $m_2 = p$, $m_3 = q$, $m_4 = -1$. (See [10, Figure 6.3].) We now see CASE 2 in [10, Theorem 6.7]. Here we note that the associated

oriented pretzel link L' (defined in [10, TYPE II.6.5]) is of type $(2m, -2, -2)$. Since $\sum_{j=1}^4 \frac{m_j}{|m_j|} = -1 + 1 + 1 - 1 = 0$ and L' is of type $(2m, -2, -2) \neq \pm(2, -2, 2)$ if $m > 1$, we check CASE 2B in [10, Theorem 6.7]. Then we see that K is fibered if and only if L' is fibered. For L' , we check CASE 1 in [10, Theorem 6.7], and verify that L' is not fibered since no n_j is ± 1 and L' is not equivalent to a pretzel link of type $\pm(2, -2, \dots, 2, -2, n)$ with an integer n . Therefore we conclude that K is not fibered. \square

The second conclusion of Proposition 3 follows from results of Ozsváth and Szabó, also achieved by using the Heegaard Floer homology theory. We first prepare the following claim, which is implicitly used in [18, Proof of Corollary 1.3].

Claim 2. *If α/β -Dehn surgery on a non-trivial knot K in S^3 yields an L-space for some coprime integers α, β with $\beta \geq 2$, then α -Dehn surgery on K also yields an L-space.*

Proof. Given coprime integers α, β and a knot K in S^3 , let $S_{\alpha/\beta}^3(K)$ denote the 3-manifold obtained from S^3 by α/β -surgery on K . We recall the following general formula given in [22, Proposition 9.5]:

$$\mathrm{rk} \widehat{HF}(S_{\alpha/\beta}^3(K)) = |\alpha| + 2 \max(0, (2\nu(K) - 1)|\beta| - |\alpha|) + |\beta| \left(\sum_s \left(\mathrm{rk} H_*(\hat{A}_s) - 1 \right) \right).$$

This holds for any pair of coprime integers α, β .

For simplicity, let $X(\nu(K), \alpha, \beta)$ denote $\max(0, (2\nu(K) - 1)|\beta| - |\alpha|)$ and Y denote $\sum_s \left(\mathrm{rk} H_*(\hat{A}_s) - 1 \right)$. Then we have

$$(1) \quad \mathrm{rk} \widehat{HF}(S_{\alpha/\beta}^3(K)) = |\alpha| + 2X(\nu(K), \alpha, \beta) + |\beta|Y.$$

Now, for some coprime integers α, β with $\beta \geq 2$, we assume that $S_{\alpha/\beta}^3(K)$ is an L-space, i.e., by definition,

$$\mathrm{rk} \widehat{HF}(S_{\alpha/\beta}^3(K)) = |\alpha|.$$

It then suffices to show that $S_{\alpha}^3(K)$ is an L-space, i.e., $\mathrm{rk} \widehat{HF}(S_{\alpha}^3(K)) = |\alpha|$.

On the other hand, in general, we see that $\mathrm{rk} \widehat{HF}(S_{\alpha}^3(K)) - |\alpha| \geq 0$ for any integer α as follows. In the proof of [20, Proposition 5.1], it is claimed that

$$\chi(\widehat{HF}(S_{\alpha}^3(K))) = |H_1(S_{\alpha}^3(K); \mathbb{Z})|.$$

Also see [21, Section 2]. By definition, the Euler characteristic (the left-hand side) is the alternating sum of the dimensions of $\widehat{HF}(S_{\alpha}^3(K))$. Hence, it is not greater than the total rank of $\widehat{HF}(S_{\alpha}^3(K))$, i.e.,

$$\mathrm{rk} \widehat{HF}(S_{\alpha}^3(K)) \geq \chi(\widehat{HF}(S_{\alpha}^3(K))) = |H_1(S_{\alpha}^3(K); \mathbb{Z})| = |\alpha|.$$

From this equation, in order to obtain $\mathrm{rk} \widehat{HF}(S_{\alpha}^3(K)) = |\alpha|$, it suffices to show that $\mathrm{rk} \widehat{HF}(S_{\alpha}^3(K)) - |\alpha| \leq 0$. Actually, we have from equation (1);

$$(2) \quad \mathrm{rk} \widehat{HF}(S_{\alpha}^3(K)) - |\alpha| = 2X(\nu(K), \alpha, 1) + Y.$$

Note here that we have $Y \leq 0$ as follows. It is seen that

$$(3) \quad 2X(\nu(K), \alpha, \beta) + |\beta|Y = 0$$

from equation (1) and the assumption that $\text{rk}\widehat{HF}(S_{\alpha/\beta}^3(K)) = |\alpha|$. Thus, together with $X(\nu(K), \alpha, \beta) \geq 0$ by definition, we have $Y \leq 0$.

If $\nu(K) \leq 0$, then

$$X(\nu(K), \alpha, 1) = \max(0, (2\nu(K) - 1) - |\alpha|) = 0$$

holds. Since $Y \leq 0$, together with equation (2), we obtain that $\text{rk}\widehat{HF}(S_{\alpha}^3(K)) - |\alpha| \leq 0$ as desired.

If $\nu(K) \geq 1$, then we have $X(\nu(K), \alpha, 1) < X(\nu(K), \alpha, \beta)$ from the assumption that $\beta \geq 2$. Thus, together with $Y \leq 0$ and equation (3), we obtain that

$$2X(\nu(K), \alpha, 1) + Y < 2X(\nu(K), \alpha, \beta) + Y = -|\beta|Y + Y \leq 0.$$

Together with equation (2), this implies that $\text{rk}\widehat{HF}(S_{\alpha}^3(K)) - |\alpha| \leq 0$ as desired. \square

Then, in [21, Corollary 1.3], Ozsváth and Szabó proved that if a knot K in S^3 admits an integral Dehn surgery yielding an L-space, then the Alexander polynomial $\Delta_K(t)$ has the form

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^k (-1)^{k-j} (t^{n_j} + t^{-n_j})$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \cdots < n_k$. This means that all non-zero coefficients of $\Delta_K(t)$ are ± 1 . \square

Remark 2. In the above proof, Claim 2 is actually necessary for the following reason. By the Cyclic Surgery Theorem established in [4], all cyclic surgeries on hyperbolic knots in S^3 are shown to be integral surgeries. However, the Finite Surgery Theorem of [3] shows that finite surgeries on hyperbolic knots in S^3 are half-integral or integral. In other words, at present, we cannot rule out the possibility of a half-integral surgery and it is currently only a conjecture that such finite surgeries are integral: See [13, Problem 1.77 A(6)] for more information.

3. CALCULATION OF THE ALEXANDER POLYNOMIALS

In this section, we prove the following proposition, which will be shown by direct calculations of the Alexander polynomials.

Proposition 4. *Let K be a pretzel knot of type $(-1, 2n, p, q)$, where n is an integer and p, q are odd positive integers with $p \leq q$. If every non-zero coefficient of the Alexander polynomial of K is ± 1 , then $n = 1$ and $p = 3$.*

Recall that the Alexander polynomial $\Delta_L(t)$ for a link L satisfies the following skein relation (see [14, pp. 82] for example):

$$(4) \quad \Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t),$$

where L_+ , L_- , and L_0 possess diagrams D_+ , D_- and D_0 which differ only in a small neighborhood as shown in Figure 2.

Remark 3. Let l be a positive integer, and $\Delta_l(t)$ the Alexander polynomial of a $(2, l)$ -torus link. Set $f_l = \sum_{i=0}^l t^i$. Then we have $\Delta_l(-t) = (-t)^{(1-l)/2} f_{l-1}$. See [12, pp. 98] for example.

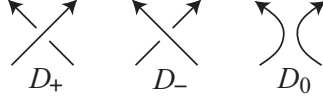


FIGURE 2. Skein triples

Proof of Proposition 4. We divide our proof of Proposition 4 into three claims. We denote by $P(a_1, \dots, a_j)$ a pretzel link of type (a_1, \dots, a_j) , and by $[g(t)]_j$ the coefficient of t^j in a polynomial $g(t)$.

Claim 3. *Let n be an integer with $n \geq 1$. Let p and q be odd integers with $3 \leq p \leq q$. Let K be a pretzel knot of type $(-1, -2n, p, q)$. Then we have*

$$[\Delta_K(t)]_1 = \begin{cases} -4 & \text{if } n = 1, \\ -3 & \text{if } n \geq 2, \end{cases}$$

where $\Delta_K(t)$ is normalized so that $\min \deg \Delta_K(t) = 0$ and $[\Delta_K(t)]_0 > 0$.

Proof. Let $K = P(-1, -2n, p, q)$ with $1 \leq n$ and $3 \leq p \leq q$. By applying the skein formula (4) at crossings in the $(-2n)$ -twists repeatedly, we can obtain a resolving tree such that each leaf node corresponds to either $P(-1, 0, p, q)$ or $P(-1, -1, p, q)$. Notice that $P(-1, 0, p, q)$ is equivalent to a connected sum of a $(2, p)$ -torus knot and a $(2, q)$ -torus knot. Then we have

$$\begin{aligned} \Delta_K(t) &= \Delta_{2n-1}(t) \Delta_{P(-1, 0, p, q)}(t) - \Delta_{2n}(t) \Delta_{P(-1, -1, p, q)}(t) \\ &= \Delta_{2n-1}(t) \Delta_p(t) \Delta_q(t) - \Delta_{2n}(t) \Delta_{P(-1, -1, p, q)}(t). \end{aligned}$$

Next we calculate the Alexander polynomial of $P(-1, -1, p, q)$ by the same argument as above. By applying the skein formula (4) at crossings in the p -twists repeatedly, we can obtain a resolving tree such that each leaf node corresponds to either $P(-1, -1, 0, q)$ or $P(-1, -1, 1, q)$. Note that $P(-1, -1, 0, q)$ is equivalent to a $(2, q)$ -torus knot and that $P(-1, -1, 1, q)$ is equivalent to a $(2, q-1)$ -torus link. Then we have

$$\begin{aligned} \Delta_{P(-1, -1, p, q)}(t) &= \Delta_{p-1}(t) \Delta_{P(-1, -1, 0, q)}(t) + \Delta_p(t) \Delta_{P(-1, -1, 1, q)}(t) \\ &= \Delta_{p-1}(t) \Delta_q(t) + \Delta_p(t) \Delta_{q-1}(t). \end{aligned}$$

Hence we have

$$\Delta_K(t) = \Delta_{2n-1}(t) \Delta_p(t) \Delta_q(t) - \Delta_{2n}(t) \Delta_{p-1}(t) \Delta_q(t) - \Delta_{2n}(t) \Delta_p(t) \Delta_{q-1}(t).$$

To calculate easily, we consider the polynomial obtained by substituting $-t$ in the Alexander polynomial, namely, $\Delta_K(-t)$. By Remark 3, we have

$$\begin{aligned} \Delta_K(-t) &= (-t)^{(4-p-q-2n)/2} (f_{2n-2} f_{p-1} f_{q-1} - f_{2n-1} f_{p-2} f_{q-1} - f_{2n-1} f_{p-1} f_{q-2}) \\ &\doteq -f_{2n-2} f_{p-1} f_{q-1} + f_{2n-1} f_{p-2} f_{q-1} + f_{2n-1} f_{p-1} f_{q-2}. \end{aligned}$$

Here the symbol \doteq means that both sides are equal up to multiplications by units of the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. Here we recall that $1 \leq n$ and $3 \leq p \leq q$.

Then we have

$$\begin{aligned} [f_{2n-2}f_{p-1}f_{q-1}]_1 &= \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n \geq 2, \end{cases} \\ [f_{2n-1}f_{p-2}f_{q-1}]_1 &= 3, \\ [f_{2n-1}f_{p-1}f_{q-2}]_1 &= 3. \end{aligned}$$

Therefore we have

$$[\Delta_K(-t)]_1 = \begin{cases} -2 + 3 + 3 = 4 & \text{if } n = 1, \\ -3 + 3 + 3 = 3 & \text{if } n \geq 2, \end{cases}$$

that is,

$$[\Delta_K(t)]_1 = \begin{cases} -4 & \text{if } n = 1, \\ -3 & \text{if } n \geq 2. \end{cases}$$

□

Claim 4. *Let n be an integer with $n \geq 2$. Let p and q be odd integers with $3 \leq p \leq q$. Let K be a pretzel knot of type $(-1, 2n, p, q)$. Then we have*

$$[\Delta_K(t)]_3 = 2,$$

where $\Delta_K(t)$ is normalized so that $\mindeg \Delta_K(t) = 0$ and $[\Delta_K(t)]_0 > 0$.

Proof. The proof is similar to that of Claim 3. Let $K = P(-1, 2n, p, q)$ with $2 \leq n$ and $3 \leq p \leq q$. By applying the skein formula (4) at crossings in the $2n$ -twists repeatedly, we can obtain a resolving tree such that each leaf node corresponds to $P(-1, 0, p, q)$ or $P(-1, 1, p, q)$. Then we have

$$\begin{aligned} \Delta_K(t) &= \Delta_{2n-1}(t)\Delta_{P(-1,0,p,q)}(t) + \Delta_{2n}(t)\Delta_{P(-1,1,p,q)}(t) \\ &= \Delta_{2n-1}(t)\Delta_p(t)\Delta_q(t) + \Delta_{2n}(t)\Delta_{P(-1,1,p,q)}(t). \end{aligned}$$

By applying the same argument as above at crossings in the p -twists, we have

$$\begin{aligned} \Delta_{P(-1,1,p,q)}(t) &= \Delta_{p-1}(t)\Delta_{P(-1,1,0,q)}(t) + \Delta_p(t)\Delta_{P(-1,1,1,q)}(t) \\ &= \Delta_{p-1}(t)\Delta_q(t) + \Delta_p(t)\Delta_{P(-1,1,1,q)}(t). \end{aligned}$$

Notice that $P(-1, 1, 1, q)$ is equivalent to a $(2, q+1)$ -torus link. By applying the skein formula (4), we have $\Delta_{q+1} = \Delta_{q-1}(t) + (t^{-1/2} - t^{1/2})\Delta_q(t)$. Hence we have

$$\begin{aligned} \Delta_K(t) &= \Delta_{2n-1}(t)\Delta_p(t)\Delta_q(t) + \Delta_{2n}(t)\Delta_{p-1}(t)\Delta_q(t) + \Delta_{2n}(t)\Delta_p(t)\Delta_{q-1}(t) \\ &\quad + (t^{-1/2} - t^{1/2})\Delta_{2n}(t)\Delta_p(t)\Delta_q(t), \end{aligned}$$

and then we have

$$\begin{aligned} \Delta_K(-t) &\doteq -tf_{2n-2}f_{p-1}f_{q-1} - tf_{2n-1}f_{p-2}f_{q-1} - tf_{2n-1}f_{p-1}f_{q-2} \\ &\quad + (1+t)f_{2n-1}f_{p-1}f_{q-1}. \end{aligned}$$

Here we recall that $2 \leq n$ and $3 \leq p \leq q$. Then we have

$$\begin{aligned}
[tf_{2n-2}f_{p-1}f_{q-1}]_3 &= [f_{2n-2}f_{p-1}f_{q-1}]_2 \\
&= 6, \\
[tf_{2n-1}f_{p-2}f_{q-1}]_3 &= [f_{2n-1}f_{p-2}f_{q-1}]_2 \\
&= \begin{cases} 5 & \text{if } p=3, q \geq 3, \\ 6 & \text{if } 5 \leq p \leq q, \end{cases} \\
[tf_{2n-1}f_{p-1}f_{q-2}]_3 &= [f_{2n-1}f_{p-1}f_{q-2}]_2 \\
&= \begin{cases} 5 & \text{if } p=3, q=3, \\ 6 & \text{if } p \geq 3, q \geq 5, \end{cases} \\
[(1+t)f_{2n-1}f_{p-1}f_{q-1}]_3 &= [f_{2n-1}f_{p-1}f_{q-1}]_3 + [f_{2n-1}f_{p-1}f_{q-1}]_2 \\
&= \begin{cases} 8+6=14 & \text{if } p=3, q=3, \\ 9+6=15 & \text{if } p=3, q \geq 5, \\ 10+6=16 & \text{if } 5 \leq p \leq q. \end{cases}
\end{aligned}$$

Therefore we have

$$[\Delta_K(-t)]_3 = \begin{cases} -6-5-5+14=-2 & \text{if } p=3, q=3, \\ -6-5-6+15=-2 & \text{if } p=3, q \geq 5, \\ -6-6-6+16=-2 & \text{if } 5 \leq p \leq q, \end{cases}$$

that is, $[\Delta_K(t)]_3 = 2$. \square

Here we note that $P(-1, 2, p, q)$ is equivalent to $P(-2, p, q)$.

Claim 5. *Let p and q be odd integers with $5 \leq p \leq q$. Let K be a pretzel knot of type $(-2, p, q)$. Then we have*

$$[\Delta_K(t)]_4 = -2,$$

where $\Delta_K(t)$ is normalized so that $\min \deg \Delta_K(t) = 0$ and $[\Delta_K(t)]_0 > 0$.

Proof. Let $K = P(-2, p, q)$ with $5 \leq p \leq q$. By applying the skein formula (4) at a crossing in the (-2) -twists, we have

$$\Delta_K(t) = \Delta_p(t)\Delta_q(t) + (t^{-1/2} - t^{1/2})\Delta_{p+q}(t).$$

Then we have

$$\Delta_K(-t) \doteq -tf_{p-1}f_{q-1} + (1+t)f_{p+q-1}.$$

Here we recall that $5 \leq p \leq q$. Then we have $[tf_{p-1}f_{q-1}]_4 = [f_{p-1}f_{q-1}]_3 = 4$ and $[(1+t)f_{p+q-1}]_4 = 1+1=2$. Therefore we have $[\Delta_K(-t)]_4 = -4+2=-2$, that is, $[\Delta_K(t)]_4 = -2$. \square

This completes the proof of Proposition 4. \square

4. PROOF OF THEOREM 1

Proof of Theorem 1. By Propositions 3 and 4, if a hyperbolic Montesinos knot K admits a non-trivial cyclic or finite surgery, then K is equivalent to a $(-1, 2, 3, q)$ -pretzel knot, where q is an odd positive integer with $3 \leq q$. This K is actually equivalent to a $(-2, 3, q)$ -pretzel knot. Then Mattman showed in [15, Theorem 1.1 and 1.2] that, among such pretzel knots, only the $(-2, 3, 7)$ - and $(-2, 3, 9)$ - can have cyclic/finite surgeries, and the surgery slopes are as described in Theorem 1. This completes the proof of Theorem 1. \square

Remark 4. The techniques we have used in this paper cannot be applied to the $(-2, 3, q)$ -pretzel knots as they are fibered and all non-zero coefficients of their Alexander polynomials are ± 1 .

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SCHOOL OF MATHEMATICS EDUCATION, NARA UNIVERSITY OF EDUCATION, TAKABATAKE-CHO,
NARA, 630-8528, JAPAN

E-mail address: `ichihara@nara-edu.ac.jp`

URL: `http://mailsrv.nara-edu.ac.jp/~ichihara/index.html`

GRADUATE SCHOOL OF SCIENCE, OSAKA CITY UNIVERSITY, OSAKA 558-8585, JAPAN

E-mail address: `jong@sci.osaka-cu.ac.jp`

URL: `http://www.ex.media.osaka-cu.ac.jp/~d07sa009/index.html`